Local Maximum Likelihood Multiuser Detection

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Abstract – The optimum multiuser detector achieves global maximum likelihood and has a complexity growing exponentially with the number of users. In this paper, we propose local maximum likelihood (LML) approach to multiuser detection with an arbitrary neighborhood size. As the neighborhood size is one, two, etc., up to the total number of users, the computational complexity of the LML detector is linear, quadratic, etc., up to exponential in the total number of users. A family of local-maximum-likelihood likelihood-ascent-search (LMLAS) detectors is then proposed, each of which is shown to achieve local maximum likelihood (LML) defined with neighborhood size one. In this paper, we propose the LML multiuser detection with any neighborhood size. A family of local-maximum-likelihood likelihood-ascent-search (LMLAS) detectors is developed that achieve the LML multiuser detection.

I. INTRODUCTION

Verdú [1][2] showed that an optimum multiuser detector in terms of maximum likelihood can achieve significant performance improvement over the conventional detector. However, its computational complexity grows exponentially with the number of active users. Many low-complexity suboptimal multiuser detectors [3] have been developed, including the suboptimal tree-type maximum-likelihood sequence detectors, linear suboptimum detectors, successive interference cancellation (SIC) detector, decorrelating decision-feedback detector (DDFD), and parallel interference cancellation (PIC) detector.

A family of local maximum likelihood (LAS) multiuser detectors was developed recently [4][5], which originated from a family of modified Hopfield neural network algorithms [6]. In the family of LAS detector, there is a set of wide-sense sequential LAS (WSLAS) detectors that achieve local maximum likelihood (LML) defined with neighborhood size one. In this paper, we propose the LML multiuser detection with any neighborhood size. A family of local-maximum-likelihood likelihood-ascent-search (LMLAS) detectors is developed that achieve the LML multiuser detection.

II. LOCAL MAXIMUM LIKELIHOOD DETECTORS

A. Received CDMA Signal

Consider a bit-synchronous Gaussian CDMA channel of $K$ users with processing gain $M$. After sampled at the chip rate, the received one-shot CDMA signal in matrix form can be written as

$$y = SAb + n$$

where $S = (s_1, …, s_k)$ is the signature waveform matrix and $A = \text{diag}(A_1, …, A_K)$ is the diagonal matrix of signal amplitudes. The $i$th user’s bit $b_i$ is spread by $s_i \in \mathbb{R}^M$, $||s_i|| = 1$. $n$ is an $M$-dimensional Gaussian random vector with zero mean and covariance matrix $\sigma^2I$. Assume the continuous-time signature waveforms have a frequency bandwidth of bit rate times processing gain. Then the $M$-dimensional random vector $y$ is a sufficient statistic of $b$. In this case, the transpose of the signature matrix $S^T$ is equivalent to a matched-filter bank. By applying $S^T$ to the data of (1), the output of the matched-filter bank is

$$r = S^Ty = RAb + z$$

where $R = S^TS$ is the crosscorrelation matrix with unit diagonal elements $R_{ii} = 1$, $z = S^Tn$ is a Gaussian random vector with zero mean and covariance matrix $\sigma^2R$.

B. Global maximum likelihood detector

Assume that all users independently transmit bits $\pm 1$’s with equal probability. From (2), a likelihood function can be written as

$$L(r | b) = \frac{1}{(2\pi\sigma^2)^{\frac{M}{2}}} |R|^{\frac{1}{2}} \exp \left( -\frac{(RAb - r)^T R^{-1}(RAb - r)}{2\sigma^2} \right)$$

where if the inverse of $R$ does not exist, $R^{-1}$ denotes its pseudo-inverse and $|R|$ is the product of nonzero eigenvalues of $R$.

It is well-known [3] that the minimization of the error probability leads to the optimum detector that selects the
Lemma 1: For \( \forall r \in \mathbb{R}^K \), \( \Psi_{j}^{LML}(r) \) satisﬁes the following inequalities:

\[
\sum_{i \in L} A_i b_i \left( r - \sum_{j \in L} R_{ij} A_{ij} b_{ij} \right) \geq 0 \tag{9}
\]

for all \( L \subseteq \{1, \ldots, K\} \) such that \( 1 \leq |L| \leq J \).

According to Lemma 1, for \( \forall b \in \Psi_{j}^{LML}(r) \), the total number of inequalities in (9) that \( b \) satisﬁes is equal to \( \sum_{i=1}^{J} |L| \). In particular, for \( b^{GML} \), the total number is \( 2^K - 1 \). This suggests the relative computational complexity for the search of an LML point with a given neighborhood size \( J \). In other words, if the neighborhood size is one, two, etc., up to \( K \), the computational complexity is linear, quadratic, etc., up to exponential in the total number of users. The GML point is speciﬁed by the most conditions.

That is, for \( \forall r \in \mathbb{R}^K \), \( b^{GML} \) is the GML point if and only if

\[
\sum_{i \in L} A_i b_i^{GML} \left( r - \sum_{j \in L} R_{ij} A_{ij} b_{ij}^{GML} \right) \geq 0 \tag{10}
\]

for all \( L \subseteq \{1, \ldots, K\} \) such that \( 1 \leq |L| \leq K \).

D. Local maximum likelihood region

Limit points of a search detector are ﬁnally demodulated vectors. The observation region of a limit point is the decision region of the detector with an according initial.

Definition 4: Let \( \Psi_{j}^{LML}(r) \subseteq \{-1, 1\}^K \) be the set of limit points of a search detector \( \phi \) with observation \( r \). The LML region of a limit point \( b \) in the observation space is deﬁned as \( V_{j}^{LML}(b) = \{ r \in \mathbb{R}^K | b \in \Psi_{j}^{LML}(r) \} \).

Property 3: For \( \forall b \in \{-1, 1\}^K \), the LML region of \( b \) with neighborhood size \( J \) is \( V_{j}^{LML}(b) = \{ r \in \mathbb{R}^K | r \in \Psi_{j}^{LML}(r) \} \).

The following results are obvious.

Property 4: For \( \forall b \in \{-1, 1\}^K \), \( V_{j}^{LML}(b) \supseteq V_{2}^{LML}(b) \supseteq \ldots \supseteq V_{K}^{LML}(b) \).

Property 5: For \( \forall b \in \{-1, 1\}^K \), \( V_{j}^{LML}(b) \) is convex.

As the neighborhood size increases, the LML region is bounded by more and more hyperplanes, and is closer and closer to the decision region \( V_{K}^{LML}(b) \) of the GML detector.

E. LML detector and local minimum error probability

The LML detectors can be deﬁned based on the deﬁnition of LML points.

Definition 6: For a detector \( b^\phi \), if \( b^\phi \in \Psi_{j}^{LML}(r) \) for \( \forall r \in \mathbb{R}^K \), \( b^\phi \) is said to be an LML detector with neighborhood size \( J \) and denoted by \( b^\phi \in \Psi_{j}^{LML} \).

\(^1\) In contrast to the local maximum likelihood (LML) detectors proposed in this paper, we call the optimum detector the global maximum likelihood (GML) detector.
In the notation similar to (4) for the GML detector, an LML detector with neighborhood size $J$ can be written as

$$b_{j}^{\text{LML}} = \arg \max_{b \in \mathcal{N}[b_{j}^{\text{L}}]} \Lambda(r | b), \forall r \in \mathbb{R}^K. \tag{11}$$

When $J = K$, (11) becomes (4) because the largest neighborhood for any vector is $\{-1, 1\}^K$.

**Property 6:** If $b^\phi \in \Psi_{J}^{\text{LML}}$, then $b^\phi \in \Psi_{I}^{\text{LML}}$ for $\forall I < J$; or conversely, if $b^\phi \not\in \Psi_{J}^{\text{LML}}$, then $b^\phi \not\in \Psi_{I}^{\text{LML}}$ for $\forall J > I$. □

Property 6 indicates that an LML detector with neighborhood size smaller than $J$ is also an LML detector with neighborhood size smaller than $J$. It is clear that the GML detector is an LML detector of any neighborhood size.

**Definition 7:** Consider two detectors $b^\phi$ and $b^\psi$. If $b^\phi(r) \in \mathcal{N}[b^\phi(r)]$ for $\forall r \in \mathbb{R}^K$, then $b^\phi$ is said to be in the neighborhood of $b^\psi$ and denoted by $b^\phi \in \mathcal{N}[b^\psi]$. □

For any detector $b^\phi$ with demodulated vector $b^\phi(r)$ for $\forall r \in \mathbb{R}^K$, an error probability $P_e(b^\phi)$ is associated. Thus, $P_e(b^\phi)$ is a function of $b^\phi$ and is determined by $b^\phi(r) \in \{-1, 1\}^K$ for all $r \in \mathbb{R}^K$. Because of the relationship between two detectors each of which is in other’s neighborhood, we can define a local minimum error probability as follows.

**Definition 8:** If $P_e(b^\phi) \leq P_e(b^\psi)$ for $\forall b^\psi \in \mathcal{N}[b^\phi]$, then $P_e(b^\phi)$ is a local minimum error probability with neighborhood size $J$.

**Property 7:** $b^\phi$ achieves a local minimum error probability with neighborhood size $J$ iff $\Pr(b^\phi \in \Psi_{J}^{\text{LML}} \mathcal{N}_J) = 1$. □

### III. A FAMILY OF LMLAS DETECTORS

#### A. A generalized LMLAS detector

In this section, we develop a family of detectors to search LML points. In the search, the likelihood ascent search is considered because of two reasons. First, every increase of likelihood will contribute to the decrease of error probability, thus the computation in the search is worthy. Second, an LML point achieves the maximum likelihood in its neighborhood. A local search with the likelihood ascent must lead to an LML point. Specifically, from one bit vector $b(t)$, the likelihood of each of other bit vectors in the neighborhood of $b(t)$ is compared with the likelihood of $b(t)$. If a bit vector in the neighborhood has a higher likelihood than $b(t)$, then the bit vector is accepted as the new bit vector $b(t+1)$ and a new search in the neighborhood of $b(t+1)$ starts. This search is repeated until the bit vector obtained achieves the local maximum likelihood in its neighborhood. To computationally efficiently compare the likelihood, the negative gradient of metric at the current bit vector shall be used. We shall call this family of search detectors the family of local maximum likelihood likelihood ascent search (LMLAS) detectors, which is defined by the following generalized LMLAS detector.

**GLMLAS detector:** Given an initial $b(0) \in \{-1, 1\}^K$, and a sequence of bit index subsets $L(t) \subseteq \{1, ..., K\}$ such that $1 \leq |L(t)| \leq J$ for all $t \geq 0$. For all $t \geq 0$, repeat the following update until termination. If

$$\sum_{j \in L(t)} b_j(t) + \sum_{j \in L(t)} W_j b_j(t) \leq 0 \tag{12}$$

where $h(t)$ is the $i$th component of the negative metric gradient $b_i(t)$ evaluated at $b(t)$, then the bits are updated by

$$b_j(t+1) = \begin{cases} -b_j(t), & \text{if } k \in L(t), \\ b_j(t), & \text{otherwise}, \end{cases} \tag{13}$$

and the negative metric gradient is updated by

$$h(t+1) = h(t) + 2 \sum_{j \in L(t)} b_j(t) W_j \tag{14}$$

where $W_k$ is the $k$th column vector of matrix $W$. If $b(t) = b'$ for $0 \leq t \leq t'$ and

$$\bigcup_{t \leq t' \in \mathbb{Z}} \left\{b' - 2 \sum_{j \in L(t)} b_j(t) \right\} = \mathcal{N}_J(b'), \tag{15}$$

then terminate with $b'$ of the finally demodulated bit vector.

$L(t)$ determines the vector to be compared with the current vector $b(t)$. If the flip condition (12) is satisfied, $b(t)$’s bits whose indices belong to $L(t)$ are flipped and a new vector $b(t+1)$ is accepted. Specifying a neighborhood size $J$ and an index subset sequence $L(t)$ for $t \geq 0$ in the GLMLAS detector, one produces an LMLAS detector. Hence, the GLMLAS detector defines a family of LMLAS detectors.

#### B. An example of LMLAS detector

Given below is an example how to specify the sequence of index subsets.

**E-LMLAS detector:** Denote $\Omega(l) \equiv \{L \subseteq \{1, ..., K\} \mid |L| = l\}$. (i) Given an initial $b(0) \in \{-1, 1\}^K$. (ii) For $j = J$ to 1, for each $L \in \Omega(j)$, update bits according to (12)-(14). (iii) If no flip occurs in step (ii), terminate; otherwise, go to (ii).

Note that the computations in the E-LMLAS detector is not optimized. A bit vector that has been compared before may be compared again, which increases computational complexity. Clearly, the E-LMLAS detector belongs to the family of the LMLAS detectors. Performance of the E-LMLAS detector is examined in simulation.

### IV. CHARACTERISTICS OF LMLAS DETECTORS

#### A. Likelihood ascent and stability
As indicated by Theorem 1, the family of LMLAS detectors are indeed likelihood ascent search (LAS) detectors.

**Theorem 1:** For \( \forall \mathbf{r} \in \mathbb{R}^K \), the GLMLAS detector guarantees monotonic likelihood ascent, i.e.,
\[
\Lambda[\mathbf{r} | \mathbf{b}(t+1)] \geq \Lambda[\mathbf{r} | \mathbf{b}(t)], \quad \forall t \geq 0,
\]
where the equality holds iff \( \mathbf{b}(t+1) = \mathbf{b}(t) \).

By repeatedly applying Theorem 1 to every search step, the following corollary is obtained.

**Corollary 1:** For \( \forall \mathbf{b}(0) \in \{-1, 1\}^K \) and \( \mathbf{r} \in \mathbb{R}^K \), let \( \mathbf{b}^{\text{GLMLAS}} \) be a fixed point of the GLMLAS detector with initial \( \mathbf{b}(0) \). (i) If \( \mathbf{b}(0) \notin \Psi^{\text{GLMLAS}}(\mathbf{r}) \), then \( \mathbf{b}^{\text{GLMLAS}} = \mathbf{b}(0) \) and \( \Lambda(\mathbf{r} | \mathbf{b}^{\text{GLMLAS}}) > \Lambda(\mathbf{r} | \mathbf{b}(0)) \); (ii) if \( \mathbf{b}(0) \in \Psi^{\text{GLMLAS}}(\mathbf{r}) \), then \( \mathbf{b}^{\text{GLMLAS}} = \mathbf{b}(0) \).

**Theorem 2:** (i) For \( \forall \mathbf{r} \in \mathbb{R}^K \), after a finite number of search steps, the GLMLAS detector converges to a fixed point; (ii) the GLMLAS detector converges to a fixed point in a finite number of steps with probability one.

**Theorem 3:** For \( \forall \mathbf{r} \in \mathbb{R}^K \), let \( \Psi^{\text{GLMLAS}}(\mathbf{r}) \) and \( \Psi^{\text{LMLAS}}(\mathbf{r}) \) be the limit point set and the fixed point set of the GLMLAS detector, respectively. Then \( \Psi^{\text{GLMLAS}}(\mathbf{r}) = \Psi^{\text{LMLAS}}(\mathbf{r}) \).

Theorem 3 implies that the GLMLAS detector never enters a limit cycle because all limit points are fixed points.

**B. Fixed point set of LMLAS detectors**

The following theorem shows that the LMLAS detectors are local maximum likelihood (LML) detectors.

**Theorem 4:** Let \( \Psi^{\text{GLMLAS}}_J(\mathbf{r}) \) be the fixed point set of GLMLAS detector with neighborhood size \( J \). Then \( \Psi^{\text{GLMLAS}}_J(\mathbf{r}) = \Psi^{\text{LML}}_J(\mathbf{r}), \quad \forall \mathbf{r} \in \mathbb{R}^K \).

Since the result in Theorem 4 is true for any observation, \( \Pr(\Psi^{\text{GLMLAS}}_J = \Psi^{\text{LML}}_J) = 1 \).

**Corollary 2:** Every LML detector with neighborhood size \( J \) achieves a local minimum error probability with the same neighborhood size \( J \).

**Corollary 3:** For any initial \( \mathbf{b}(0) \in \{-1, 1\}^K \), \( \mathbf{b}^J_{\text{GLMLAS}} \in \Psi^{\text{LML}}_J \).

As indicated by Corollary 3, every LMLAS detector with neighborhood size \( J \) is an LML detector with the same neighborhood size. Because of this, all characteristics of an LML point and its LML region are characteristics of the fixed point of an LMLAS detector.

**Property 8:** For \( \forall \mathbf{r} \in \mathbb{R}^K \), \( \Psi_1^{\text{GLMLAS}}(\mathbf{r}) \supseteq \Psi_2^{\text{GLMLAS}}(\mathbf{r}) \supseteq \ldots \supseteq \Psi_\infty^{\text{GLMLAS}}(\mathbf{r}) \).

**C. Monotonic descent of error probability**

The monotonic likelihood ascent of the GLMLAS detector directly results in the monotonic descent of error probability.

**Theorem 5:** The GLMLAS detector monotonically reduces error probability,
\[
P_r[\mathbf{b}(t+1)] \leq P_r[\mathbf{b}(t)], \quad \forall t \geq 0
\]
where the equality holds iff \( \Pr[\mathbf{b}(t+1) = \mathbf{b}(t)] = 1 \).

**Theorem 6:** Let \( \mathbf{b}^J_{\text{GLMLAS}} \) be an LMLAS detector with an arbitrary neighborhood size \( J \) with initial \( \mathbf{b}(0) \). Then \( P_r(\mathbf{b}^J_{\text{GLMLAS}}) \leq P_r[\mathbf{b}(0)] \)
\[
(18)
\]
where \( P_r(\mathbf{b}^J_{\text{GLMLAS}}) \) is a local minimum error probability with neighborhood size \( J \) and the equality holds iff \( \Pr[\mathbf{b}(0) \in \Psi^J_{\text{LML}}] = 1 \).

In Theorem 6 the initial \( \mathbf{b}(0) \) of the LMLAS detector can be any detector of any kind. A local minimum error probability achieved by an LML detector may not be necessarily smaller than an error probability achieved by a non-LML detector. However, as we see from Theorem 6, unless a non-LML detector differs from an LML detector only on observations of zero probability measure, its error probability can be reduced by a followed LMLAS detector to a local minimum error probability with the same neighborhood size defined by the LMLAS detector.

Almost all well-known detectors (except the GML detector), including conventional detector, decorrelating detector, MMSE detector, SIC, and PIC detector, are not LML detectors even with the smallest neighborhood size, size one. Due to Property 6, this means that none of these detectors is an LML detector with any neighborhood size. Thus, the error probabilities of these detectors can be reduced to local minima by a followed LMLAS detector. This has been verified in our simulations. Part of the simulation results for the conventional detector, decorrelating detector, and the MMSE receiver as the initial is reported in this paper. On the other hand, we note that due to Theorem 4 and Property 8, an LMLAS detector can not improve any LMLAS (or LML) detector with the same or a larger neighborhood size.

**V. CONCLUSIONS**

We extend the concept of LML detection to an arbitrary neighborhood size, and develop a family of LMLAS detectors achieving LML detection with any neighborhood size.

When the neighborhood size is equal to one, two, etc., up to the total number of users, the complexity of an LML detector is linear, quadratic, etc., up to exponential in the number of users, respectively.
As the neighborhood size increases, the number of LML points decreases and the difference between an LML region and the optimum (GML) decision region reduces while the complexity for decision of an LML point increases.

The LMLAS detectors guarantee monotonic likelihood ascent in search, thus guaranteeing convergence to a fixed point in a finite number of search steps with probability one. Any fixed point of an LMLAS detector with a given neighborhood size is an LML point with the same neighborhood size, and vice versa. An LMLAS detector with a given neighborhood size monotonically reduces error probability step by step and finally reaches a local minimum error probability defined with the same neighborhood size. The error probability of any detector can be reduced to a local minimum by a followed LMLAS detector unless with probability one this detector itself is an LML detector with the same or larger neighborhood size. Since the conventional detector, all linear detectors including the decorrelating detector and MMSE detector, the SIC detector, and the PIC detectors all are not LML detectors even with neighborhood size one, their error probabilities can be reduced to local minimums by a followed LMLAS detector. As the neighborhood size of an LMLAS detector increases, its error probability decreases and computational complexity increases. One can make a tradeoff between error performance and computational complexity by choosing a neighborhood size in an LMLAS detector. The BER floor at high SNR may exist when the neighborhood size is small. However, the BER floor is at a lower level, starts to appear at a higher SNR, and finally disappears as the neighborhood size of the LMLAS detector becomes larger and finally becomes the total number of users.

REFERENCES

Fig. 1. Processing gain is $M = 11$, SNR = 11 dB, initial is the conventional detector, and the system is perfectly power-controlled. The BER of the LMLAS detector decreases as its neighborhood size increases.

BER of the LMLAS-J detector with $J = 6$ is overlapped with that of the GML detector. The BER of the LMLAS detector decreases as its neighborhood size increases. Confirming Theorem 6, the LMLAS detectors improve the initial conventional detector.

Fig. 2. The condition is the same as in Fig. 1. The computation of the GML detector is optimized. The computational complexity of an LMLAS detector increases as its neighborhood size increases, which presents the relative complexity of LML detectors in Lemma 1.

Fig. 4. The conditions are the same as in Fig. 3 except that the initial is the decorrelating detector. Confirming Theorem 6, the LMLAS detectors improve the initial decorrelating detector.

Fig. 3. Processing gain is $M = 15$; number of users is $K = 10$; system is perfectly power-controlled. The initial is the conventional detector. The BER of the LMLAS-J detector with $J = 6$ is overlapped with that of the GML detector. The BER of the LMLAS detector decreases as its neighborhood size increases. Confirming Theorem 6, the LMLAS detectors improve the initial conventional detector.

Fig. 5. The conditions are the same as in Fig. 3 except that the initial is the MMSE detector. Confirming Theorem 6, the LMLAS detectors improve the initial MMSE detector.